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The Structural Operator F and Its Role in the Theory of Retarded Systems* I

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1. INTRODUCTION

In recent years the retarded linear functional differential equations (FDE) have been often investigated by using the C_0-semigroups on the product space $\mathbb{R}^n \times L^2$. This has been motivated by the great flexibility this space offers in studies of many problems, especially in the control theory. Among the recent publications involving studies of retarded equations by using the space $\mathbb{R}^n \times L^2$ we quote Banks and Burns [1] and Delfour [5]. Many other references along with historical accounts can be found in those papers. For the theory of retarded equations using space C of continuous functions the reader is referred to Hale [8].

Although the semigroup approach offers a general view on problems involving functional differential equations, many useful special features of FDEs are lost in the semigroup representation. On the other hand, going back to a detailed

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representation of FDEs quickly leads to cumbersome calculations. To retain some of the essential features of retarded FDEs in the general semigroup representation, the present authors were led to introduce, in 1975, a certain operator F. This operator is a bounded linear mapping from $\mathbb{R}^n \times L^2$ into itself and depends on the structure of the retarded part of the FDE. It has been since called either the "structural" or the "hereditary" operator F. Initially introduced to alleviate the burden of a cumbersome notation in systems with many delays, this operator soon turned out to be an extremely useful device in the analysis of FDEs. A preliminary account of some of the useful properties of F was given in Bernier and Manitius [3], Delfour and Manitius [7], Delfour *et al.* [6], and Manitius [14]. It was noticed that many important qualitative properties of retarded FDEs are related to the operator F. For instance, the well-known bilinear form described by Hale [8, Chaps. 6, 7] corresponds in the setting of $\mathbb{R}^n \times L^2$ to a bilinear form generated by F. Another property is that the solution of a retarded FDE is zero for all $t \geqslant 0$ if and only if the initial data are in the null space of F. The operator F has some nice intertwining properties with the adjoint and the transposed semigroups. It also plays an important role in the problems of controllability, observability (Manitius [14]), completeness of eigenfunctions (Section 5 of this paper and Manitius [15]), and the operator Riccati equation (cf. Delfour *et al.* [6]).

This motivated the present authors to investigate more deeply the operator F and see how it relates to the various semigroups in $\mathbb{R}^n \times L^2$ constructed from the autonomous retarded FDEs. Previously, we have defined F for a class of equations with finite number of delays and an integral term. In this paper we generalize F to the class of FDEs in which the right-hand side is a Lebesgue–Stieltjes integral of the segment of past solution, with the kernel which is an arbitrary $n \times n$ matrix function of bounded variation. Fundamental properties of F are then systematically examined. In Section 2 we investigate conditions for F to be one to one, onto or with dense image. It is then shown that the adjoint of F is intertwined in a certain way with the adjoint and transposed generators and semigroups. If the null space of F is not just $\{0\}$, it is shown in Section 3 that the semigroup can be redefined on the quotient space $\mathbb{R}^n \times L^2$ by the null space of F. The questions as to when the usual and the quotient space semigroups are one to one for all t are answered. In Part II the role of F in the spectral theory is examined. It is shown in Section 4 that the spectral projections can be expressed by operators involving the bilinear form generated by F. There are also some interesting relationships between the eigenspaces of the transposed and the adjoint generators that involve F. The resolvent operator is also simply expressed as a composition of F with other operators. In Section 5 the simple forms for the resolvent are used to obtain the criteria of completeness of eigenfunctions associated with the FDE. These criteria are stated in the complex domain and constitute a generalization of earlier results by Levinson and McCalla [13] which were obtained for scalar equations only. Section 5 is complementary to another study of completeness given in Manitius [15], in which the operator F also plays a prominent role.

Various results contained in this paper have been obtained by the authors during the period 1975–1978. Some of them were even announced and used in earlier publications (e.g., Theorem 4.4(iii) and Corollary 5.4. are quoted in Manitius [14]), although the complete proofs appear in this paper for the first time. To streamline the presentation, some of the more technical proofs have been deferred to the Appendix.

Notation and Terminology

Let $\mathbb{R}$ and C denote the set of real and complex numbers, respectively; the complex conjugate of λ in C will be denoted by $\bar{\lambda}$. Let $\mathbb{R}^n$ (resp. C^n) be the real (resp. complex) Hilbert space of dimension n ($n \geqslant 1$, an integer) endowed with the usual Euclidean norm $|\cdot|$ and scalar product $x \cdot y$. The transpose (a_{ji}) and adjoint $(\bar{a}_{ji})$ of an $n \times n$ matrix $A = (a_{ij})$ defined on C will be denoted by A^{T} and A^*, respectively. The space of Lebesgue measurable maps $[a, b] \to \mathbb{R}^n$ which are square integrable (resp. essentially bounded) will be denoted by $L^2[a, b]$ (resp. $L^\infty[a, b]$) and its norm will be denoted by $\| \|_2$ (resp. $\| \|_\infty$). The inner product in $L^2[a, b]$ will be denoted by $(\, , \,)_2$. The Sobolev space of functions $f\colon [a, b] \to \mathbb{R}^n$ with a derivative Df in $L^2[a, b]$ will be denoted by $H^1[a, b]$ and its norm by $\| \|_{H^1}$. The space of continuous functions $[a, b] \to \mathbb{R}^n$ will be denoted by $C[a, b]$ and its norm by $\| \|_C$. Occasionally when the values of the functions lie in a space different of $\mathbb{R}^n$, e.g., $\mathbb{R}^m$ for $m \neq n$, we shall write $L^2(a, b; \mathbb{R}^m)$, $L^\infty(a, b; \mathbb{R}^m)$, or $H^1(a, b; \mathbb{R}^m)$. The space of functions of bounded variation $\eta\colon [a, b] \to \mathbb{R}^n$ (resp. $\mathbb{R}^n \times \mathbb{R}^n$) will be denoted by $BV[a, b]$ and their total variation $V(\eta, [a, b])$. Given $h > 0$, let M^2 denote the product space $\mathbb{R}^n \times L^2[-h, 0]$ endowed with the inner product

$$((\phi, \psi)) = \phi^0 \cdot \psi^0 + (\phi^1, \psi^1)_2, \tag{1.1}$$

where each element ϕ of M^2 is identified with a pair (ϕ^0, ϕ^1), $\phi^0 \in \mathbb{R}^n$, $\phi^1 \in L^2[-h, 0]$. Given two Banach spaces X and Y, we denote by $\mathscr{L}(X, Y)$ the Banach space of all continuous linear maps $L\colon X \to Y$ endowed with the natural norm. When $X = Y$, we write $\mathscr{L}(X)$ in place of $\mathscr{L}(X, X)$; the identity in $\mathscr{L}(X)$ is denoted by I. The dual of an element L in $\mathscr{L}(X, Y)$ will be denoted by L^* (in $\mathscr{L}(Y^*, X^*)$). When X and Y are Hilbert spaces we shall say that L^* viewed as an element of $\mathscr{L}(Y, X)$ is the adjoint of L. The symbols $\operatorname{Ker} L$ and $\operatorname{Im} L$ will denote the null space and the image of L, respectively. The closure in the norm topology of a subset S of a Banach space will be denoted by $\bar{S}$. Finally, the derivative of a function $f\colon [a, b] \to \mathbb{R}^n$ will be denoted $\dot{f}$, Df or df/dt.

2. The Structural Operator F

This section is divided into two parts. In the first one we study the basic properties of the operator F; in the second one we characterize $\operatorname{Ker} F$ and $\operatorname{Im} F$.

2.1. *Definition and Basic Properties*

In this section we define the operator F for a general class ot autonomous linear retarded equations given by

$$\frac{dx}{dt}(t) = \int_{[-h,0]} d\eta(\theta)\, x(t+\theta), \qquad t \geqslant 0,$$
$$x(\theta) = \phi(\theta) \qquad \text{in } [-h, 0], \tag{2.1}$$

where $\eta(\cdot)$ is an $n \times n$ matrix of real *functions of bounded variation*, i.e., $\eta(\cdot) \in BV$. We shall exhibit the role of F in the dependence of solutions on the initial data. We shall also obtain the adjoint F^* of F and explain the role of F in the hereditary product of Hale [8, p. 173].

If we assume that η is of the form

$$\eta(\theta) = A_0\chi_{[0]}(\theta) + \sum_{i=1}^{N} A_i\chi_{]-h_i,0]}(\theta) + \int_{-h}^{\theta} A_{01}(\alpha)\, d\alpha, \tag{2.2}$$

where χ_I denotes the characteristic function of the set I, A_i are $n \times n$ matrices, $A_{01}(\cdot) \in L^1(-h, 0; \mathbb{R}^{n\times n})$, and

$$0 = h_0 < h_1 < \cdots < h_i < h_{i+1} < \cdots < h_N = h \tag{2.3}$$

is an ordered finite sequence of real numbers, then (2.1) reduces to

$$\frac{dx}{dt}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t-h_i) + \int_{-h}^{0} A_{01}(\alpha)\, x(t+\alpha)\, d\alpha. \tag{2.4}$$

The matrix η corresponding to (2.4) is not unique. It could also be chosen as

$$\eta(\theta) = -A_0\chi_{]-\infty,0[}(\theta) - \sum_{i=1}^{N} A_i\chi_{]-\infty,-h_i]}(\theta) - \int_{\theta}^{0} A_{01}(\alpha)\, d\alpha. \tag{2.5}$$

For initial functions in $C[-h, 0]$, and $\eta(\cdot) \in BV[-h, 0]$ Eq. (2.1) has a unique solution which continuously depends on ϕ.

In order to exhibit the role of the initial function ϕ and the solution x: $[0, \infty[\rightarrow \mathbb{R}^n$, let us rewrite (2.1) in the following form

$$\begin{aligned} \frac{dx(t)}{dt} &= \int_{[-t,0]} d\eta(\theta)\, x(t+\theta) + \int_{[-h,-t[} d\eta(\theta)\, \phi(t+\theta), \qquad 0 \leqslant t \leqslant h, \\ &= \int_{[-h,0]} d\eta(\theta)\, x(t+\theta), \qquad t > h. \end{aligned} \tag{2.6}$$

The term containing ϕ is zero for $t \geqslant h$. This term very naturally suggests the introduction of the linear map $\phi \rightarrow H\phi$ given by

$$(H\phi)(\alpha) = \int_{[-h,\alpha[} d\eta(\theta)\, \phi(\theta - \alpha), \qquad \alpha \in [-h, 0]. \tag{2.7}$$

For each $\phi \in C[-h, 0]$, $(H\phi)(\alpha)$ is well defined for each $\alpha \in [-h, 0]$, and $H\phi$ is a function of bounded variation on $[-h, 0]$. Moreover

$$|(H\phi)(\alpha)| \leqslant \int_{[-h,\alpha[} |\, d\eta(\theta)|\, |\phi(\theta-\alpha)| \leqslant V(\eta, [-h, 0[) \cdot \|\phi\|_C, \qquad \alpha \in [-h, 0]. \tag{2.8}$$

This shows that H is a continuous linear map from $C[-h, 0]$ to $L^2[-h, 0]$. We will now show that the domain of H can be extended to $L^2[-h, 0]$ without losing the continuity of H.

Introduce the following (closed) subspaces of $C[-h, 0]$, endowed with the sup norm

$$C_0 = \{\phi \in C[-h, 0] \mid \phi(0) = 0\}, \qquad C_h = \{\phi \in C[-h, 0] \mid \phi(-h) = 0\}. \tag{2.9}$$

THEOREM 2.1. *Let $\eta \in BV[-h, 0]$. Then*

(i) *H has a continuous extension (still denoted by H)*

$$H: L^2[-h, 0] \rightarrow L^2[-h, 0]. \tag{2.10}$$

(ii) *The restriction $\hat{H}$ of H to C_0, $\phi \rightarrow \hat{H}\phi = H\phi$, $\phi \in C_0$, is a continuous linear map $\hat{H}$: $C_0 \rightarrow C_h$.*

Proof.[1] (i) Take arbitrary $\phi \in C[-h, 0]$, $\psi \in L^2[-h, 0]$. Since $H\phi \in L^2[-h, 0]$, we can write

$$(\psi, H\phi)_2 = \int_{-h}^{0} d\alpha\, \psi(\alpha) \cdot \int_{[-h,\alpha[} d\eta(\theta)\, \phi(\theta - \alpha). \tag{2.11}$$

[1] The first author is pleased to acknowledge discussions with R. B. Vinter of the Imperial College of Science and Technology, London, England for part (i) of the proof of Theorem 2.1.

Since the integral $\int_{-h}^{0} d\alpha \mid \psi(\alpha)| \int_{[-h,\alpha[} \mid d\eta(\theta)| \mid \phi(\theta - \alpha)|$ is bounded, we can use Fubini's theorem (cf. Rudin [14, Theorem 7.8]) to change the order of integration. We have

$$\begin{aligned} |(\psi, H\phi)_2| &= \left| \int_{-h}^{0} d\eta^*(\theta) \int_{\theta}^{0} d\alpha\, \psi(\alpha) \cdot \phi(\theta - \alpha) \right| \\ &\leqslant \int_{-h}^{0} | d\eta^*(\theta)| \int_{\theta}^{0} d\alpha \mid \psi(\alpha)| \mid \phi(\theta - \alpha)| \leqslant c \| \psi \|_2 \| \phi \|_2 , \end{aligned} \tag{2.12}$$

where $c = V(\eta^*, [-h, 0])$. By taking $\psi = H\phi$ we obtain $\| H\phi \|_{L^2} \leqslant c \| \phi \|_{L^2}$. This and the density of continuous functions in L^2 prove that H has a continuous extension, still denoted by H, defined on all of $L^2[-h, 0]$.

(ii) See Appendix. ∎

Remark 2.1. Theorem 2.1(i) shows that the function $\theta \to (H\phi)(\theta)$ with $\phi \in L^2[-h, 0]$ is a good L^2-"function," although it is not necessarily well defined for all $\theta \in [-h, 0]$. The point values of $(H\phi)(\theta)$ do not matter as long as we treat $H\phi$ as an element of $L^2[-h, 0]$. Observe that $\alpha \to (H\phi)(\alpha)$ is indistinguishable in $L^2[-h, 0]$ from the function

$$\alpha \to \int_{[-h,\alpha]} d\eta(\theta)\, \phi(\theta - \alpha), \tag{2.13}$$

which differs from (2.7) on a set of measure zero. For instance, when η is given by (2.2) and ϕ is continuous

$$(H\phi)(\alpha) = \sum_{i=1}^{N} A_i \phi(-h_i - \alpha)\, \chi_{]-h_i,0]}(\alpha) + \int_{-h}^{\alpha} A_{01}(\theta)\, \phi(\theta - \alpha)\, d\theta \tag{2.14}$$

while

$$\begin{aligned} \int_{[-h,\alpha]} d\eta(\theta)\, \phi(\theta - \alpha) &= A_0 \phi(0)\, \chi_{[0]}(\alpha) + \sum_{i=1}^{N} A_i \phi(-h_i - \alpha)\, \chi_{[-h_i,0]}(\alpha) \\ &\quad + \int_{-h}^{\alpha} A_{01}(\theta)\, \phi(\theta - \alpha)\, d\theta. \end{aligned} \tag{2.15}$$

In some computations where $(H\phi)(\alpha)$ is defined everywhere, it will be important to distinguish between definitions (2.7) and (2.13).

Remark 2.2. Theorem 2.1 is also true with $L^p[-h, 0]$, $1 \leqslant p < \infty$, in place of $L^2[-h, 0]$.

With the above definition, for $\phi \in C[-h, 0]$ Eq. (2.1) on $[0, T]$ for all $T > 0$ now takes the form

$$\frac{dx}{dt}(t) = (\mathscr{A}x)(t) + \begin{Bmatrix} (H\phi)(-t), & 0 \leqslant t \leqslant h \\ 0, & t > h \end{Bmatrix}, \qquad t \in [0, T], \quad x(0) = \phi(0), \tag{2.16}$$

where $\mathscr{A}$ is a continuous linear map from $C[0, T]$ into $L^2[0, T]$, given by the first term on the right-hand side of Eq. (2.6). In view of (2.16), one can interpret the contribution of the initial function ϕ on $[-h, 0]$ to the solution $x(t)$, $t \geqslant 0$, as that of the "forcing function" $f(t) = (H\phi)(-t)\chi_{[0,h]}$.

We now observe that, by Theorem 2.1(i), for initial data in M^2, i.e., $x(0) = \phi^0$, $\phi(\theta) = \phi^1(\theta)$ on $[-h, 0[$ with $\phi^1 \in L^2[-h, 0]$, the "forcing function" $f(t) = (H\phi)(-t)\chi_{[0,h]}$ is in $L^2[0, h]$, and that Eq. (2.16) still makes sense a.e. on $[0, T]$.

PROPOSITION 2.2. *Given $\phi = (\phi^0, \phi^1)$ in $M^2 = \mathbb{R}^n \times L^2[-h, 0]$, the equation*

$$\begin{gathered} \frac{dx}{dt}(t) = \int_{[-h,0]} d\eta(\theta)\, x(t + \theta), \qquad \text{a.e. in } [0, \infty[, \\ x(0) = \phi^0, \quad x(\theta) = \phi^1(\theta), \qquad \theta \in [-h, 0[, \quad \eta \in BV[-h, 0], \end{gathered} \tag{2.17}$$

has a unique solution in $H^1[0, T]$ and for all $T > 0$

$$\| x \|_{H^1[0,T]} \leqslant c(T) \| \phi \|_{M^2}, \qquad \| \phi \|_{M^2} = \{ | \phi^0 |^2 + \| \phi^1 \|_2^2 \}^{1/2}, \tag{2.18}$$

where $c(T) > 0$ is a constant which solely depends on T.

The proof of the above proposition uses the previous theorem and standard arguments; hence it will be omitted. This result was obtained by Vinter [16, 17] using semigroup techniques. It shows that the three extra conditions imposed by Borisovic and Turbabin [4] are always verified.

The interpretation of $H\phi$ as a "forcing function" yields the following observation.

PROPOSITION 2.3. *The solution $x(\cdot)$ of (2.17) is identically zero in $[0, T]$ for $T \geqslant h$ if and only if $\phi^0 = 0$ and $H\phi^1 = 0$.*

We now *define* the operator F in $\mathscr{L}(M^2)$ by

$$F\phi = (\phi^0, H\phi^1). \tag{2.19}$$

F can be interpreted as a matrix of operators $\left(\begin{smallmatrix} I & 0 \\ 0 & H \end{smallmatrix}\right)$ on the product $\mathbb{R}^n \times L^2$. We refer to F as the *structural operator* associated with system (2.1) or (2.17).

Remark 2.3. If we pick $[\phi]$ in the quotient space $M^2/\operatorname{Ker} F$, the map $[\phi] \to x(\cdot; \phi)$ is well defined, linear, continuous, and injective.

We now associate with F the following *bilinear form* on M^2

$$\langle \psi, \phi \rangle = ((\psi, F\phi)). \tag{2.20}$$

In general this pairing is not symmetrical and does not separate points (unless $F^* = F$ and F is injective with a dense image in M^2). For ϕ and ψ in the subspace C of M^2,

$$C = \{(\phi(0), \phi) \in M^2 \mid \phi \in C[-h, 0]\}, \tag{2.21}$$

the pairing (2.20) is equal to

$$\langle \psi, \phi \rangle = \psi(0) \cdot \phi(0) + \int_{-h}^{0} d\alpha\, \psi(\alpha) \cdot \int_{[-h,\alpha[} d\eta(\theta)\, \phi(\theta - \alpha). \tag{2.22}$$

By Fubini's theorem (η is a matrix of real finite measures), we can change the order of integration since the integrand is bounded and the product measure $d\eta(\cdot) \times d\alpha$ is finite:

$$\langle \psi, \phi \rangle = \psi(0) \cdot \phi(0) + \int_{-h}^{0} d\xi \int_{[\theta,0[} \psi(\theta - \xi) \cdot d\eta(\theta)\, \phi(\xi). \tag{2.23}$$

In this form the pairing (2.20) coincides with the one introduced by Hale [8, p. 173] with the difference that he defined it for ϕ in $C[-h, 0]$ and ψ in $C[0, h]$.

Remark 2.4. In many computations we shall use the fact that the map H (resp. F) defined on $C[-h, 0]$ can be extended to the larger space $L^2[-h, 0]$ (resp. M^2). Usually the computation will be performed with ϕ in $C[-h, 0]$; then we shall somehow assert that the result is true for all ϕ in M^2.

The dual operators H^* and F^* of H and F will also naturally arise in the study of "adjoint systems." *It is very fundamental to note that they have the same structure as the operators H and F.* The dual operator F^* of F is of the form

$$F^*\psi = (\psi^0, H^*\psi^1), \tag{2.24}$$

where H^* is the dual operator of H. Now, by using Fubini's theorem, for ϕ and ψ in $C[-h, 0]$

$$\begin{aligned}
(H\phi, \psi)_2 &= \int_{-h}^{0} d\alpha \int_{[-h,\alpha[} d\eta(\theta)\, \phi(\theta - \alpha) \cdot \psi(\alpha) \\
&= \int_{[-h,0[} d\eta(\theta) \int_{\theta}^{0} d\alpha\, \phi(\theta - \alpha) \cdot \psi(\alpha) \\
&= \int_{[-h,0[} d\eta(\theta) \int_{\theta}^{0} d\xi\, \phi(\xi) \cdot \psi(\theta - \xi) \\
&= \int_{-h}^{0} d\xi \int_{[-h,\xi[} d\eta(\theta)\, \phi(\xi) \cdot \psi(\theta - \xi) \\
&= \int_{-h}^{0} d\xi\, \phi(\xi) \cdot \int_{[-h,\xi[} d\eta^*(\theta)\, \psi(\theta - \xi) = (\phi, H^*\psi)_2\,,
\end{aligned} \tag{2.25}$$

where $\eta^*(\theta)$ is the transposed matrix of $\eta(\theta)$, and for all ψ in $C[-h, 0]$

$$(H^*\psi)(\alpha) = \int_{[-h,\alpha[} d\eta^*(\theta)\,\psi(\theta-\alpha); \tag{2.26}$$

by techniques used in the proof of Theorem 2.1, (2.26) can now be extended to a unique continuous map H^* defined on all of $L^2[-h, 0]$. As a result H^* *is identical to H up to a transposition of the matrix* $\eta(\theta)$. It is naturally associated with the analog of Eq. (2.17), which can be formally written as

$$\begin{aligned} &\frac{dp}{dt}(t) = \int_{[-h,0]} d\eta^*(\theta)\,p(t+\theta) \qquad \text{a.e. in } [0,\infty[,\\ &p(0) = \psi^0, \quad p(\theta) = \psi^1(\theta) \qquad \text{in } [-h, 0[, \qquad \psi = (\psi^0, \psi^1) \in M^2. \end{aligned} \tag{2.27}$$

Remark 2.5. If we pick $[\psi]$ in $M^2/\operatorname{Ker} F^*$, the map $[\psi] \to p(\cdot\,;\psi)$ is well defined, linear, continuous, and injective.

2.2. *Characterizations of the Null Space and Range of F*

The special structure of the operator F can be used to characterize situations where F or F^* are injective or surjective, or have a dense image in M^2. We begin with simple observations. It is obvious that

$$\operatorname{Ker} F = \{0\} \times \operatorname{Ker} H, \qquad \operatorname{Im} F = \mathbb{R}^n \times \operatorname{Im} H.$$

Consequently, we will investigate $\operatorname{Ker} H$ and $\operatorname{Im} H$. Consider the two simple examples

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \tag{2.28}$$

$$\dot{x}(t) = \int_{-h}^{0} A_{01}(\theta)\, x(t+\theta)\, d\theta. \tag{2.29}$$

In the first case $(H\phi)(\alpha) = A_1\phi(-h-\alpha)$. Let A_1 have rank $m \leqslant n$. Then

$$\operatorname{Ker} H = \{\phi \in L^2[-h, 0] \mid A_1\phi(\alpha) = 0 \text{ a.e. in } [-h, 0]\}, \tag{2.30}$$

and

$$\operatorname{Ker} H = \{0\} \Leftrightarrow A_1 \text{ is invertible } (m = n).$$

In this case $\operatorname{Im} H$ is isomorphic with $L^2(-h, 0; \mathbb{R}^m)$ and, therefore, is closed for any m. Furthermore, if $m = n$, H is surjective. In the second case, $(H\phi)(\alpha) = \int_{-h}^{\alpha} A_{01}(\theta)\,\phi(\theta-\alpha)\,d\theta$ is a convolution. If $A_{01}(\cdot)$ is in $L^2(-h, 0; \mathbb{R}^{n\times n})$ then the function $\alpha \to (H\phi)(\alpha)$ is continuous (cf. Hewitt and Stromberg [11, Theorem 21.33]), and $H\phi \in C_h$. In this case $\operatorname{Im} H$ is clearly not closed in $L^2[-h, 0]$, but it might be dense, depending on $A_{01}(\cdot)$.

PROPOSITION 2.4. *A necessary condition for* $\overline{\operatorname{Im} H^{L^2}} = L^2[-h, 0]$ *is that for all* $\alpha > -h$, *the map*

$$\phi \to [\hat{H}\phi](\alpha) \colon C_0 \to \mathbb{R}^n \tag{2.32}$$

be surjective.

Proof. See Appendix. ▮

Note that if condition (2.32) holds for some $\bar{\alpha} > -h$, then it also holds for all $\alpha > \bar{\alpha}$. Indeed, if the expression

$$[\hat{H}\phi](\bar{\alpha}) = \int_{[-h,\bar{\alpha}[} d\eta(\theta)\, \phi(\theta - \bar{\alpha})$$

can be made equal to an arbitrary vector x of $\mathbb{R}^n$ by picking some ϕ in C_0, then for any $\alpha \geqslant \bar{\alpha}$ we define $\psi(\theta) = 0$ for $\bar{\alpha} - \alpha \leqslant \theta \leqslant 0$ and $\psi(\theta) = \phi(\theta - \bar{\alpha} + \alpha)$ for $-h \leqslant \theta < \bar{\alpha} - \alpha$. By construction ψ belongs to C_0 and

$$[\hat{H}\psi](\alpha) = \int_{[-h,\alpha[} d\eta(\theta)\, \psi(\theta - \alpha) = \int_{[-h,\bar{\alpha}[} d\eta(\theta)\, \phi(\theta - \bar{\alpha}) = x.$$

The above considerations show that condition (2.32) need only be verified in a small neighborhood of the point $-h$. *This indicates that the density of* $\operatorname{Im} F$ *in* M^2 *necessarily depends on the behavior of* η *in the vicinity of* $-h$. We summarize "local conditions" in the next proposition.

PROPOSITION 2.5. *Each of the following conditions is equivalent to condition* (2.32) *of Proposition* 2.4:

$$\exists \bar{\alpha} > -h,\ \forall \alpha \in\,]-h, \bar{\alpha}[,\ \phi \to [\hat{H}\phi](\alpha) \text{ is surjective}; \tag{2.33}$$

$$\exists \bar{\alpha} > -h,\ \forall \alpha \in\,]-h, \bar{\alpha}[,\ \nexists y \neq 0; \text{ in } \mathbb{R}^n, \text{ s.t. } V(\eta^*(\cdot)y, [-h, \alpha[) = 0, \tag{2.34}$$

where $V(\eta^*(\cdot)y, [-h, \alpha[)$ *is the total variation in* $[-h, \alpha[$ *of the function of bounded variation* $\theta \to \eta^*(\theta)y$.

Proof. (2.32) $\Leftrightarrow$ (2.33), from remarks preceding the proposition. (2.33) $\Leftrightarrow$ (2.34). The surjectivity condition (2.33) is equivalent to

$$\nexists y \neq 0 \text{ in } \mathbb{R}^n, \qquad \forall \phi \in C_0, \qquad y \cdot [\hat{H}\phi](\alpha) = 0. \tag{2.35}$$

But

$$\forall \phi \in C_0, \qquad \int_{[-h,\alpha[} d\eta^*(\theta) y \cdot \phi(\theta - \alpha) = 0 \Leftrightarrow V(\eta^*(\cdot)y, [-h, \alpha[) = 0. \tag{2.36}$$

This is sufficient to establish the equivalence of (2.33) and (2.34). ▮

Remark 2.6. Note that none of the conditions (2.32), (2.33), or (2.34) uses the behavior of η at the point 0. For instance, when η is given by (2.2) the properties of $\hat{H}$ and H are completely independent of A_0.

Again Proposition 2.5 indicates that properties of operators H and F are completely characterized by the structure of η in a neighborhood $[-h, -h+\tau]$, $\tau > 0$, of $-h$. It is then natural to extend the class of systems characterized by η in (2.2).

DEFINITION 2.6. We say that the $n \times n$ matrix η of functions of bounded variation has an *isolated atom at* $-h$ if there exists a neighborhood $[-h, -h+\tau]$, $\tau > 0$, of $-h$ where η is of the form

$$\eta(\theta) = A_h \chi_{]-h,-h+\tau]}(\theta) + \int_{-h}^{\theta} A_{01}(\alpha)\, d\alpha, \tag{2.37}$$

where A_h is an arbitrary $n \times n$ matrix and $\alpha \to A_{01}(\alpha)$ is an arbitrary $n \times n$ matrix of functions in $L^2(-h, -h+\tau; \mathbb{R})$.

In the above definition $\eta(-h) = 0$ and η experiences a jump of height A_h from $-h$ to $-h^+$.

Note that $\eta(\cdot)$ given by (2.2) satisfies Definition 2.6 with $A_h = A_N$ and $\tau = h - h_{N-1}$. Since the definition imposes no restriction on the behavior of $\eta(\cdot)$ in $]-h+\tau, 0]$, in the latter interval $\eta(\cdot)$ might be of a form more general than (2.2), e.g., it may have an infinite number of jump points with an accumulation point in the interval $]-h+\tau, 0]$. However, $-h$ cannot be such an accumulation point. We are now ready to specialize Proposition 2.5 to η with an isolated atom at $-h$.

COROLLARY 2.7. *Assume that η has an isolated atom at $-h$.*

(i) *A necessary condition for* $\overline{\operatorname{Im} F} = M^2$ *is*

$$\forall \alpha \in [-h, -h+\tau[, \quad \nexists y \neq 0 \text{ in } \mathbb{R}^n, \qquad A_h^* y = 0, \quad \text{and} \quad A_{01}^*(\theta) y = 0, \quad \text{a.e. in } [-h, \alpha[. \tag{2.38}$$

(ii) *A necessary condition for* $\operatorname{Ker} F = \{0\}$ *is*

$$\forall \alpha \in]-h, -h+\tau[, \nexists y \neq 0 \textit{ in } \mathbb{R}^n, A_h y = 0, \textit{and } A_{01}(\theta) y = 0, \quad \textit{a.e. in } [-h, \alpha[. \tag{2.39}$$

Proof. We only prove (i). The proof of (ii) is similar to (i) if we notice that $\overline{\operatorname{Im} F^*} = M^2$ if and only if $\operatorname{Ker} F = \{0\}$. For α in $]-h, -h+\tau[$ and η given by (2.37)

$$\begin{aligned} \eta^*(\theta) y &= A_h^* y + \int_{-h}^{\theta} A_{01}^*(\beta) y \, d\beta, \qquad && -h < \theta \leqslant \alpha, \\ &= 0, && \theta = -h, \end{aligned} \tag{2.40}$$

and by inspection

$$V(\eta^*(\cdot)y, [-h, \alpha[) = | A_h^* y | + \int_{-h}^{\alpha} | A_{01}^*(\beta) y | \, d\beta. \tag{2.41}$$

In view of the above computation, it is readily seen that

$$V(\eta^*(\cdot)y, [-h, \alpha[) = 0 \Leftrightarrow \begin{cases} A_N^* y = 0, \\ A_{01}^*(\beta) y = 0, \qquad \text{a.e. in } [-h, \alpha[. \end{cases} \tag{2.42}$$

This concludes the proof of the corollary. ▮

COROLLARY 2.8. *Assume that η has an isolated atom at $-h$. If $A_h = 0$, then a necessary condition for $\overline{\operatorname{Im} F} = M^2$ is that the $1 \times n$ vector functions given by rows of $A_{01}(\theta)$ are linearly independent on any interval $[-h, \alpha]$ for all $\alpha \in]-h, -h + \tau]$.*

The condition given by Proposition 2.4 is not sufficient. To see this let us focus on Corollary 2.7(i), which is equivalent to Proposition 2.4 under the hypothesis that $\eta(\cdot)$ has an isolated atom at $-h$. Suppose that $A_h = 0$, and that $\det A_{01}(-h) = 0$, but $\det A_{01}(\theta) \neq 0$ for all $\theta > -h$. Then (2.38) is still satisfied; however, one can construct examples such that $\operatorname{Ker} H^* \neq \{0\}$. The following example is a generalization of an example communicated to the authors by Bartosiewicz.

EXAMPLE 2.1. Let $a_1(\theta)$, $a_2(\theta)$ be two real functions in $H^1[-h, 0]$ such that

(i) $a_1(-h) = a_2(-h) = 0$,

(ii) $A_{01}(\theta) \overset{\text{df}}{=} \begin{bmatrix} \dfrac{d}{d\theta} a_1(\theta) & \dfrac{d}{d\theta} a_2(\theta) \\ a_1(\theta) & a_2(\theta) \end{bmatrix}$ is nonsingular for $\theta > -h$.

One can, for instance, take for a_i any two polynomials of different degrees, both having a zero at $-h$ (e.g., $a_1(\theta) = (\theta + h)^2/2$, $a_2(\theta) = (\theta + h)^3/6$ as suggested by Bartosiewicz). Then the null space of H^* is nontrivial because if one takes any nonzero real function x in $H^1[-h, 0]$ with $x(0) = 0$ and defines the vector function $\psi(\theta)$ by $\psi^\top(\theta) = [x(\theta), (d/d\theta)\, x(\theta)]$, then

$$[H^*\psi](\alpha) = \int_{-h}^{\alpha} A_{01}^\top(\theta)\, \psi(\theta - \alpha)\, d\theta = \int_{-h}^{\alpha} \frac{d}{d\theta} \begin{bmatrix} a_1(\theta)\, x(\theta - \alpha) \\ a_2(\theta)\, x(\theta - \alpha) \end{bmatrix} d\theta \equiv 0. \tag{2.43}$$

The above example shows that the linear independence of the rows of $A_{01}(\theta)$ is not enough to guarantee that $\overline{\operatorname{Im} F} = M^2$. We point out some stronger conditions which are sufficient but not always necessary to have $\overline{\operatorname{Im} F} = M^2$, and/or $\operatorname{Ker} F = \{0\}$.

THEOREM 2.9. *Assume that η has an isolated atom at $-h$. Then*

$$H \text{ (resp. } F) \textit{ is surjective} \Leftrightarrow A_h \textit{ is invertible.} \tag{2.44}$$

If, in addition, A_{01} is zero in a neighborhood of $-h$, then

$$\overline{\operatorname{Im} H} = L^2[-h, 0] \Leftrightarrow A_h \textit{ invertible} \Leftrightarrow \operatorname{Im} H = L^2[-h, 0]. \tag{2.45}$$

Proof. See Appendix. ▮

COROLLARY 2.10. *Assume that η has an isolated atom at $-h$. Then*

$$H \text{ (resp. } F) \textit{ has a continuous inverse} \Leftrightarrow A_h \textit{ is invertible.} \tag{2.46}$$

If, in addition, A_{01} is zero in a neighborhood of $-h$, then

$$\begin{Bmatrix} \overline{\operatorname{Im} H} = L^2[-h, 0] \\ \operatorname{Ker} H = \{0\} \end{Bmatrix} \Leftrightarrow A_h \textit{ is invertible} \Leftrightarrow H \textit{ invertible.} \tag{2.47}$$

Proof. From Theorem 2.9 and

$$A_h \text{ invertible} \Leftrightarrow A_h^* \text{ invertible} \Leftrightarrow \operatorname{Im} F^* = M^2 \Rightarrow \operatorname{Ker} F = \{0\}.$$

From the Banach inverse theorem, F has a continuous inverse. ▮

The condition "A_h invertible" is sufficient to have $\overline{\operatorname{Im} F} = M^2$ but not necessary when A_{01} is not zero in a neighborhood of $-h$. In fact, for systems with η given by (2.2) with $A_i = 0$, $i = 0,..., N$, one can still have $\overline{\operatorname{Im} F} = M^2$ if, e.g., $A_{01}(\cdot)$ is continuous and differentiable and $\det A_{01}(-h) \neq 0$. Consider the condition $\operatorname{Ker} F^* = \{0\}$ which reduces to the condition that the Volterra integral equation

$$\int_{-h}^{\alpha} A_{01}^*(\theta)\, \phi(\theta - \alpha)\, d\theta = 0, \qquad \alpha \in [-h, 0],$$

has null solution only. This equation can be rewritten as

$$\int_{-h-\alpha}^{0} A_{01}^*(\alpha + s)\, \phi(s)\, ds = 0, \qquad \alpha \in [-h, 0]. \tag{2.48}$$

If $A_{01}^*(\alpha)$ is differentiable (say, in $H^1[-h, 0]$), then by differentiating (2.48) we obtain

$$A_{01}^*(-h)\, \phi(-h - \alpha) + \int_{-h-\alpha}^{0} \left[\frac{d}{d\alpha} A_{01}^*(\alpha + s)\right] \phi(s)\, ds = 0. \tag{2.49}$$

From this equation we conclude that $\det A_{01}(-h) \neq 0$ is a sufficient condition for $\operatorname{Ker} H = \{0\}$ and $\operatorname{Ker} H^* = \{0\}$, because the invertibility of $A_{01}(-h)$ makes of (2.49) a system of homogeneous Volterra integral equations of the second kind, whose only solution is $\phi \equiv 0$.

As a consequence of Theorem 2.9 we observe that for differential-difference equations (η given by (2.2) with $A_{01} \equiv 0$) the necessary and sufficient condition for both $\operatorname{Ker} F = \{0\}$ and $\operatorname{Im} F = M^2$ is $\det A_N \neq 0$.

Note that if A_{01} is zero, at least in a neighborhood of $-h$, then (2.47) shows that

$$\operatorname{Ker} F = \{0\} \Leftrightarrow \operatorname{Ker} F^* = \{0\}.$$

Several examples indicate that this relation is also true for some systems with $A_{01}(\theta) \neq 0$, $\theta \in [-h, -h + \delta]$ (for some $\delta > 0$), but we do not know whether the relation is true in general.

Remark 2.7 (Closedness of $\operatorname{Im} F$). We terminate this section with a remark on when $\operatorname{Im} F$ is closed. We have not obtained so far a complete set of necessary and sufficient conditions. If we limit our attention to systems with $\eta(\cdot)$ given by (2.2), then the following facts are known to us presently.

1. If $N = 1$ and $A_{01} = 0$ then $\operatorname{Im} F$ is closed. This is easy to see, because $(H\phi)(\theta) = A_1\phi(-h - \theta)$, and the range of H is $L^2([-h, 0], \operatorname{Im} A_1)$, which is a closed subspace of $L^2[-h, 0]$.

2. If $N = 1$ and $\operatorname{Im} A_{01}(\theta) \subset \operatorname{Im} A_1$ $\forall \theta \in [-h, 0]$, then the above observation is still valid.

3. For N arbitrary and $A_{01}(\cdot) = 0$ one can show that $\operatorname{Im} H$ is closed if the delays h_i are all commensurable. It is not clear whether this carries through to a noncommensurable case (one can prove that it does for $N = 2$).

3. Semigroups Corresponding to Retarded Systems and the Operator F

In this section we briefly review basic facts about the semigroups in M^2 corresponding to retarded systems, and point out the role played by the operator F, especially in the adjoint semigroup. We also construct a semigroup on the quotient space $M^2/\operatorname{Ker} F$ and study its basic properties.

3.1. *Semigroups $\mathscr{S}$, $\mathscr{S}^\top$, and $\mathscr{S}^*$*

Let $\mathscr{S} = \{S(t): t \geqslant 0\}$ denote the usual C_0-semigroup in $\mathscr{L}(M^2)$ associated with the retarded system (2.17), that is $S(t)$ is given by

$$S(t)\phi = (x(t), x_t), \tag{3.1}$$

where $x(t)$ and $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$, denote the solution of (2.17) with the initial data $\phi \in M^2$. The properties of this semigroup are well known (cf. Borisovic and Turbabin [4], Vinter [15], Banks and Burns [1] and Bernier and Manitius [3]). The infinitesimal generator A of $\mathscr{S}$ is given by

$$\mathscr{D}(A) = \{(\phi(0); \phi) \in M^2 \mid \phi \in H^1[-h, 0]\}, \tag{3.2}$$

$$A\phi = (L\phi; \dot{\phi}), \qquad \text{where} \quad L\phi = \int_{[-h,0]} d\eta(\theta)\, \phi(\theta). \tag{3.3}$$

We note that for ϕ in $C[-h, 0]$

$$L\phi = \int_{[0]} d\eta(\theta)\, \phi(0) + (H\phi)(0). \tag{3.4}$$

Along with system (2.1) we consider the transposed system (2.27). Let $\mathscr{S}^\top = \{S^\top(t): t \geqslant 0\}$ denote the C_0-semigroup in $\mathscr{L}(M^2)$ corresponding to (2.27), and let $A^\top$ denote its infinitesimal generator. Obviously $\mathscr{D}(A^\top) = \mathscr{D}(A)$, and $A^\top\psi = (L^\top\psi, \dot{\psi})$, where $L^\top\psi = \int_{[-h,0]} d\eta^*(\theta)\, \psi(\theta)$.

Finally, by identifying the elements of the topological dual of M^2 with those of M^2, we have that the adjoint semigroup $\mathscr{S}^* = \{S(t)^*: t \geqslant 0\}$, where $S(t)^*$ is the adjoint operator of $S(t)$, is a C_0-semigroup in $\mathscr{L}(M^2)$. It can be shown by a straightforward computation analogous to that of Vinter [15] that the infinitesimal generator A^* of $S(t)^*$ is characterized by

$$\begin{aligned} &\mathscr{D}(A^*) = \{(\psi^0, \psi^1) \mid \psi^1 = H^*\hat{\psi}^0 + g, \text{ for some } g \in H^1[-h, 0], g(-h) = 0\}, \\ &[A^*\psi]^0 = L^\top\hat{\psi}^0 + g(0), \qquad [A^*\psi]^1 = -\dot{g}, \end{aligned} \tag{3.5}$$

where $\hat{\psi}^0$ denotes the constant function in $[-h, 0]$ equal to ψ^0

$$\hat{\psi}^0(\theta) = \psi^0, \qquad -h \leqslant \theta \leqslant 0. \tag{3.6}$$

The following theorem plays a fundamental role in the subsequent development of this paper. It shows that the operator F^* provides the essential connection between the adjoint and the transposed semigroups and between their respective infinitesimal generators (cf. Corollary 3.3). This fact has far-reaching consequences in the spectral theory of retarded FDEs. The theorem is an extension of the results of Bernier and Manitius [3, Theorem 5.4] to the case of an arbitrary η of bounded variation.

THEOREM 3.1. (i) $F^*\mathscr{D}(A^\top) \subset \mathscr{D}(A^*)$. (ii) $A^*F^* = F^*A^\top$ *on* $\mathscr{D}(A^\top)$. (iii) *For all* $t \geqslant 0$, $S(t)^*F^* = F^*S^\top(t)$.

The proof of Theorem 3.1 necessitates the following lemma, which is an extension of Theorem 2.1(ii).

LEMMA 3.2. *If we still denote by H the restriction of the map H to $H_0^1[-h, 0] = C_0[-h, 0] \cap H^1[-h, 0]$, then the map*

$$H: H_0^1[-h, 0] \to H_h^1[-h, 0],$$

where $H_h^1[-h, 0] = C_h[-h, 0] \cap H^1[-h, 0]$, is still well-defined linear and continuous when $H_0^1[-h, 0]$ and $H_h^1[-h, 0]$ are endowed with the H^1-norm. Moreover, $DH\phi = -HD\phi$, where $D\phi$ denotes the derivative of ϕ.

Proof. See Appendix. ∎

Proof of Theorem 3.1. (i) Pick ϕ in $\mathscr{D}(A^\top)$ and define $\psi = F^*\phi = (\phi(0), H^*\phi)$. Consider the expression ($\hat{\psi}^0$ and $\hat{\phi}(0)$ are defined by (3.6))

$$(\psi^1 - H^*\hat{\psi}^0)(\alpha) = (H^*\phi)(\alpha) - (H^*\hat{\phi}(0))(\alpha) = \int_{[-h,\alpha[} d\eta^*(\theta)[\phi(\theta-\alpha) - \phi(0)].$$

The function $\theta \to \phi(\theta) - \phi(0)$ belongs to $H_0^1[-h, 0]$. By Lemma 3.2 its image is equal to an element g of $H_h^1[-h, 0]$, that is $g \in H^1[-h, 0]$, $g(-h) = 0$. This shows that $F^*\phi$ belongs to $\mathscr{D}(A^*)$. (ii) Using ϕ and g as defined in (i), consider

$$\begin{aligned}[A^*F^*\phi]^1(\alpha) &= -\dot{g}(\alpha) = -\frac{d}{d\alpha}\int_{[-h,\alpha[} d\eta^*(\theta)[\phi(\theta-\alpha) - \phi(0)] \\ &= (H^*D\phi)(\alpha) = [F^*A^\top\phi]^1(\alpha),\end{aligned}$$

$$\begin{aligned}[A^*F^*\phi]^0 &= L^*\hat{\phi}(0) + g(0) = \int_{[-h,0]} d\eta^*(\theta)\,\phi(0) + \int_{[-h,0[} d\eta^*(\theta)[\phi(\theta) - \phi(0)] \\ &= \int_{[-h,0]} d\eta^*(\theta)\,\phi(\theta) = [A^\top\phi]^0 = [F^*A^\top\phi]^0.\end{aligned}$$

This establishes (ii). Part (iii) follows from (i) and (ii) by application of a general result on intertwined semigroups (cf. Bernier and Manitius [3, Lemma 5.3]). ∎

COROLLARY 3.3. (i) $\forall\phi, \psi \in \mathscr{D}(A) = \mathscr{D}(A^\top)$, $\langle\phi, A\psi\rangle = \langle A^\top\phi, \psi\rangle$. (ii) $\forall\phi, \psi \in M^2$, $\langle\phi, S(t)\psi\rangle = \langle S^\top(t)\phi, \psi\rangle$.

The above result is the classical relationship (cf. Hale [8, Sect. 7.3]) between the semigroup $\{S(t)\}$ and the transposed semigroup $\{S^\top(t)\}$ through the hereditary pairing (2.20).

Remark 3.1. Since by (2.24), (2.26) F and F^* differ from each other by a transposition of $\eta(\cdot)$, by defining $S^\top(t)^*$ as the adjoint of $S^\top(t)$ we have

$$S^\top(t)^*F = FS(t) \qquad \forall t \geq 0. \tag{3.7}$$

Remark 3.2. Any element $\psi \in M^2$ of the form $\psi = (\psi^0, 0)$ satisfies $\psi = F^*\psi$. Therefore $S(t)^*\psi = S(t)^*F^*\psi = F^*S^\top(t)\psi$ for such ψ.

Remark 3.3. From Proposition 2.3 it follows that

$$S(h)\phi = 0 \Leftrightarrow F\phi = 0. \tag{3.8}$$

In general $S(t)\phi = 0$ for some $t > h$ does not necessarily imply that $\phi \in \operatorname{Ker} F$. In fact, there are examples of retarded systems such that for some ϕ one has $S(t)\phi = 0$ for $t = nh$ but $S(t)\phi \neq 0$ for $t < nh$ (cf. Henry [9]); such a ϕ obviously does not belong to $\operatorname{Ker} F$.

A question of interest is when $\operatorname{Ker} S(t) = \{0\}$, $\forall t \geqslant 0$, that is when $\bigcup_{t>0} \operatorname{Ker} S(t) = \{0\}$. We have this result

PROPOSITION 3.4. $\bigcup_{t>0} \operatorname{Ker} S(t) = \{0\} \Leftrightarrow \operatorname{Ker} F = \{0\}$.

Proof. From (3.8) we have that $\operatorname{Ker} S(h) = \{0\} \Leftrightarrow \operatorname{Ker} F = \{0\}$. But $\operatorname{Ker} S(h) = \{0\}$ implies, via the semigroup property, that $\operatorname{Ker} S(t) = \{0\}$ for all $t \in [0, h]$ ($\operatorname{Ker} S(t) \subseteq \operatorname{Ker} S(h)$ for all $t \leqslant h$). Since for any $t > h$, say $t = lh + \sigma$, l an integer, $\sigma \in [0, h[$, $S(t) = (S(h))^l\, S(\sigma)$, we have that $\operatorname{Ker} S(h) = \{0\} \Rightarrow \operatorname{Ker} S(t) = \{0\}$, $t \geqslant h$, hence for all $t \geqslant 0$. The implication $\bigcup_{t>0} \operatorname{Ker} S(t) = \{0\} \Rightarrow \operatorname{Ker} S(h) = \{0\}$ is obvious. ∎

3.2. *Semigroup on the Quotient Space* $M^2/\operatorname{Ker} F$

In Section 2.1 we have seen that all the initial functions in $\operatorname{Ker} F$ produce the null solution of Eq. (2.1). This shows that initial functions are distinguishable from the solutions $x(t)$, $t \geqslant 0$, only modulo $\operatorname{Ker} F$. Consequently, in order to group together the initial functions that yield the same solution we introduce the quotient space $M^2/\operatorname{Ker} F$. We will show that the semigroup can be redefined on $M^2/\operatorname{Ker} F$, and that interesting characterization of the adjoint semigroup is obtained. We note that, by transposition, everything below can be repeated for the semigroup $\mathscr{S}^\perp$ redefined on the quotient space $M^2/\operatorname{Ker} F^*$.

Let $[\phi]$ denote an element of $M^2/\operatorname{Ker} F$. The dual space $(M^2/\operatorname{Ker} F)'$ is isometrically isomorphic to $(\operatorname{Ker} F)^\perp = \overline{\operatorname{Im} F^*}$. Let $[\cdot, \cdot]$ denote the duality pairing between $(M^2/\operatorname{Ker} F)'$ and $M^2/\operatorname{Ker} F$. Introduce the canonical surjection

$$\phi \to [\phi]; \qquad M^2 \to M^2/\operatorname{Ker} F \tag{3.9}$$

and the isometric isomorphism (cf. Horvath [12, p. 262])

$$\psi \to \Lambda\psi\colon \overline{\operatorname{Im} F^*} \to (M^2/\operatorname{Ker} F)' \tag{3.10}$$

such that $\forall \psi \in \overline{\operatorname{Im} F^*}$, $\forall \phi \in [\phi] \in M^2/\operatorname{Ker} F$ one has $[\Lambda\psi, [\phi]] = ((\psi, \phi))$.

PROPOSITION 3.5. (i) $\phi \in \operatorname{Ker} F \Rightarrow \forall t \geqslant 0$, $S(t)\phi \in \operatorname{Ker} F$. (ii) *The mapping* $[S(t)]$ *defined by*

$$[S(t)][\phi] = [S(t)\phi] \tag{3.11}$$

belongs to $\mathscr{L}(M^2/\operatorname{Ker} F)$.

Proof. (i) From (3.7) we have $FS(t)\phi = S^{\mathrm{T}}(t)^* F\phi = 0$ for all $t \geqslant 0$ and ϕ in $\operatorname{Ker} F$.

(ii) We first show that $[S(t)]$ is a well-defined mapping from $M^2/\operatorname{Ker} F$ into itself. If $[\phi_1] = [\phi_2] \in M^2/\operatorname{Ker} F$, then $\phi_1 - \phi_2 \in \operatorname{Ker} F$. By (i) $S(t)\phi_1 - S(t)\phi_2 = S(t)(\phi_1 - \phi_2) \in \operatorname{Ker} F$, hence $[S(t)\phi_1] = [S(t)\phi_2]$, so that $[S(t)][\phi_1] = [S(t)][\phi_2]$. The linearity of $[S(t)]$ is obvious. To prove boundedness let $\|\cdot\|_Q$ denote the quotient norm in $M^2/\operatorname{Ker} F$; for all $\phi \in [\phi]$

$$\|[S(t)][\phi]\|_Q = \|[S(t)\phi]\|_Q \leqslant \| S(t)\phi \| \leqslant c \| \phi \|.$$

Hence $\|[S(t)][\phi]\|_Q \leqslant c \inf_{\phi \in [\phi]} \| \phi \| = c \|[\phi]\|_Q$. ∎

We shall next show that the family $\{[S(t)]: t \geqslant 0\}$ is a C_0-semigroup on $M^2/\operatorname{Ker} F$. Since the space involved are still reflexive Banach spaces, the dual semigroup of $[S(t)]$, $[S(t)]^*$, is well defined on all of $(M^2/\operatorname{Ker} F)'$. Let $S_F^*(t)$ denote the restriction of $S(t)^*$ to $\overline{\operatorname{Im} F^*}$.

THEOREM 3.6. (i) *The family* $[\mathscr{S}] = \{[S(t)]: t \geqslant 0\}$ *is a* C_0*-semigroup on the product space* $M^2/\operatorname{Ker} F$. *Its domain* $\mathscr{D}([A])$ *coincides with* $[\mathscr{D}(A)]$ *and its infinitesimal generator* $[A]$ *is equal to*

$$[A][\phi] = [A\phi], \qquad \phi \in [\phi] \cap \mathscr{D}(A). \tag{3.12}$$

(ii) *The family* $\mathscr{S}_F^* = \{S_F^*(t): t \geqslant 0\}$ *is a* C_0*-semigroup on* $\overline{\operatorname{Im} F^*}$.

(iii) $S_F^*(t) = \Lambda^{-1}[S(t)]^*\Lambda$, $t \geqslant 0$.

Proof. See Appendix. ∎

As stated in Proposition 3.4, $S(t)$ is one-to-one for all $t \geqslant 0$ if and only if $\operatorname{Ker} F = \{0\}$. The same question also naturally arises for the quotient semigroup $[S(t)]$.

LEMMA 3.7. $[S(t)]$ *is one-to-one for all* $t \geqslant 0$ *if and only if*

$$\forall t \geqslant 0, \qquad S(t)\phi = 0 \Rightarrow \phi \in \operatorname{Ker} F. \tag{3.13}$$

Proof. Let (3.13) hold. We have $0 = [S(t)][\phi] = [S(t)\phi]$ implies $S(t)\phi \in \operatorname{Ker} F$. From this and (3.8) it follows that $S(h)\, S(t)\phi = 0$, or $S(t+h)\phi = 0$. By using (3.13) we have $\phi \in \operatorname{Ker} F$, or $[\phi] = 0$ in $M^2/\operatorname{Ker} F$. Hence $[S(t)]$ is one-to-one. Conversely, suppose that (3.13) is violated, that is $\exists t_0$ and $\psi \notin \operatorname{Ker} F$

such that $S(t_0)\psi = 0$. We have $[S(t_0)][\psi] = [S(t_0)\psi] = 0$ but $[\psi] \neq 0$, and $[S(t)]$ is not one-to-one for all t. ∎

Remark 3.4. Condition (3.13) is equivalent to F^*-completeness of the generalized eigenfunctions of $A^\top$; see Manitius [15]. Note that the condition $\operatorname{Ker} F = \{0\}$ (equivalent of $\operatorname{Ker} S(t) = \{0\}$, $\forall\, t \geqslant 0$) corresponds, via results of Section 5, to M^2-completeness of the generalized eigenfunctions of $A^\top$.

We terminate this section with two additional results.

LEMMA 3.8. *The hereditary product* $\langle \cdot, \cdot \rangle$ *generates the pairing* $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ *on* $M^2/\operatorname{Ker} F^* \times M^2/\operatorname{Ker} F$ *given by*

$$\langle\!\langle [\psi]^*, [\phi] \rangle\!\rangle = \langle \psi, \phi \rangle, \qquad \psi \in [\psi]^*, \phi \in [\phi], \tag{3.14}$$

which separates points.

Proof. We show that

$$\psi_2 - \psi_1 \in \operatorname{Ker} F^* \qquad \text{and} \qquad \phi_2 - \phi_1 \in \operatorname{Ker} F \Rightarrow \langle \psi_2, \phi_2 \rangle = \langle \psi_1, \phi_1 \rangle.$$

By direct computation

$$\langle \psi_2, \phi_2 \rangle = ((\psi_2, F\phi_2)) = ((\psi_2, F\phi_1)) = ((F^*\psi_2, \phi_1)) = ((F^*\psi_1, \phi_1)) = \langle \psi_1, \phi_1 \rangle.$$

Now if

$$\forall [\phi] \in M^2/\operatorname{Ker} F, \qquad \langle\!\langle [\psi]^*, [\phi] \rangle\!\rangle = 0,$$

then for $\psi \in [\psi]^*$

$$\forall \phi \in M^2, \qquad 0 = ((\psi, F\phi)) = ((F^*\psi, \phi)) \Rightarrow \psi \in \operatorname{Ker} F^* \Rightarrow [\psi] = 0.$$

This shows that $M^2/\operatorname{Ker} F$ separates points of $M^2/\operatorname{Ker} F^*$. The proof of the converse assertion is similar. ∎

Let $[S^\top(t)]$ denote the semigroup on the quotient space $M^2/\operatorname{Ker} F^*$, analogous to $[S(t)]$.

PROPOSITION 3.9. *For all* $[\psi]^*$ *in* $M^2/\operatorname{Ker} F^*$ *and* $[\phi]$ *in* $M^2/\operatorname{Ker} F$

$$\langle\!\langle [\psi]^*, [S(t)][\phi] \rangle\!\rangle = \langle\!\langle [S^\top(t)][\psi]^*, [\phi] \rangle\!\rangle. \tag{3.15}$$

Proof. From (3.14) and the following identities

$$\langle\!\langle [\psi]^*, [S(t)][\phi] \rangle\!\rangle = \langle\!\langle [\psi]^*, [S(t)\phi] \rangle\!\rangle = \langle\!\langle \psi, S(t)\phi \rangle\!\rangle$$

and

$$\langle\!\langle [S^\top(t)][\psi]^*, [\phi] \rangle\!\rangle = \langle\!\langle [S^\top(t)\psi]^*, [\phi] \rangle\!\rangle = \langle S^\top(t)\psi, \phi \rangle. \quad ∎$$

Appendix to Section 2

Proof of Theorem (2.1)(ii). Let $\phi \in C_0$. We will show that $\hat{H}$ maps C_0 into C_h. $\hat{H}$ is defined by $\hat{H}\phi = H\phi$, $\phi \in C_0$. Let $\beta \geqslant \alpha \geqslant -h$ (for $\alpha \geqslant \beta \geqslant -h$ the proof is similar). We have

$$\begin{aligned}(H\phi)(\beta) - (H\phi)(\alpha) &= \int_{[-h,\beta[} d\eta(\theta)\,\phi(\theta-\beta) - \int_{[-h,\alpha[} d\eta(\theta)\,\phi(\theta-\alpha) \\ &= \int_{[\alpha,\beta[} d\eta(\theta)\,\phi(\theta-\beta) + \int_{[-h,\alpha[} d\eta(\theta)[\phi(\theta-\beta) - \phi(\theta-\alpha)].\end{aligned} \tag{1}$$

Let $V(\eta, [a, b])$ denote the total variation of η on $[a, b]$. From (1)

$$\begin{aligned}|(H\phi)(\beta) - (H\phi)(\alpha)| \leqslant{}& V(\eta, [\alpha, \beta]) \max_{\theta\in[\alpha,\beta]} |\phi(\theta-\beta)| \\ &+ V(\eta[-h, \alpha[) \max_{\theta\in[-h,\alpha[} |\phi(\theta-\beta) - \phi(\theta-\alpha)|.\end{aligned} \tag{2}$$

By continuity of ϕ, $\forall\epsilon > 0\ \exists\delta > 0$ such that $|s - t| < \delta \Rightarrow |\phi(t) - \phi(s)| < \epsilon$. Let $|\beta - \alpha| < \delta$. Since $\phi(0) = 0$, $|\phi(\theta - \beta)| < \epsilon$ for all $\theta \in [\alpha, \beta]$. Hence

$$|(H\phi)(\beta) - (H\phi)(\alpha)| \leqslant V(\eta, [-h, 0]) \cdot \epsilon \qquad \text{if} \quad |\beta - \alpha| < \delta. \tag{3}$$

This shows that the function $\theta \to (H\phi)(\theta)$ is right-continuous on $[-h, 0[$. Similarly we show that the function is left-continuous. From (2) it also follows that if $\alpha = -h$, $\beta \downarrow \alpha$, then $(H\phi)(\beta) \to (H\phi)(-h) = 0$. Hence $\theta \to (\hat{H}\phi)(\theta) = (H\phi)(\theta)$ is continuous, and $\hat{H}\phi \in C_h$. In view of (2.8), $\hat{H}$ is continuous. ∎

Proof of Proposition 2.4. By contradiction. Assume that there exists $\bar{\alpha} > -h$ *such that*

$$\phi \to [\hat{H}\phi](\bar{\alpha}) \tag{1}$$

is not surjective. Thus there exists $y \in \mathbb{R}^n$, $y \neq 0$, such that

$$y \cdot [\hat{H}\phi](\bar{\alpha}) = 0, \qquad \forall\phi \in C_0,$$

$$0 = y \cdot \int_{[-h,\bar{\alpha}[} d\mu(\theta)\,\phi(\theta - \bar{\alpha}), \Rightarrow \int_{[-h,\bar{\alpha}[} d\mu^*(\theta) y \cdot \phi(\theta - \bar{\alpha}) = 0 \tag{2}$$

$$\Rightarrow d\mu^*(\theta) y = 0 \qquad \text{in } [-h, \bar{\alpha}].$$

We now construct

$$\begin{aligned}\psi(\alpha) &= y, \qquad \alpha \in [-h, \bar{\alpha}], \\ &= 0, \qquad \text{otherwise},\end{aligned} \tag{3}$$

and compute

$$(\psi, H\phi)_2 = \int_{-h}^{\bar{\alpha}} d\alpha\, y \cdot \int_{-h}^{\alpha} d\mu(\theta)\, \phi(\theta - \alpha) = \int_{-h}^{\bar{\alpha}} d\alpha \int_{-h}^{\alpha} d\mu^*(\theta) y \cdot \phi(\theta - \alpha) = 0. \tag{4}$$

This contradicts the density of $\operatorname{Im} H$ in $L^2[-h, 0]$ since we have constructed $\psi \neq 0$, which is orthogonal to $\operatorname{Im} H$.

Proof of Theorem 2.9. ($\Rightarrow$) We first show that in both cases "A_h invertible" is a necessary condition. By hypothesis for all α in $[-h, h + \tau[$

$$(H\phi)(\alpha) = A_h\phi(-h - \alpha) + \int_{-h}^{\alpha} A_{01}(\theta)\, \phi(\theta - \alpha)\, d\theta. \tag{1}$$

A necessary condition for $\overline{\operatorname{Im} H} = L^2[-h, 0]$ (cf. Proposition 2.5(iv)) is that for all α in an arbitrary small neighborhood of $-h$ the map $\phi \to (\hat{H}\phi)(\alpha)$ be surjective. When A_{01} is zero in a neighborhood of $-h$, this map is surjective for each α in this neighborhood if and only if A_h is invertible. When A_{01} is not zero we notice that the second term on the right-hand side of (1) is continuous for α in $[-h, -h + \tau]$ and can only fill up $L^2[-h, -h + \tau]$ in a dense way. As for the first term it can only fill up a closed proper subspace of $L^2[-h, -h + \tau]$ if A_h is singular. As a result, the invertibility of A_h is necessary to have $\operatorname{Im} H = L^2[-h, 0]$.

($\Leftarrow$) Given ψ in $L^2[-h, 0]$ and A_h invertible, we now construct ϕ in $L^2[-h, 0]$ such that $H\phi = \psi$, that is we solve the equation $H\phi = \psi$. For $\alpha \in [-h, -h + \tau[$ we have

$$\begin{aligned}[][H\phi](\alpha) &= A_h\phi(-h - \alpha) + \int_{-h}^{\alpha} A_{01}(\theta)\, \phi(\theta - \alpha)\, d\theta \\ &= A_h\phi(-h - \alpha) + \int_{-h-\alpha}^{0} A_{01}(\xi + \alpha)\, \phi(\xi)\, d\xi\end{aligned}$$

or, letting $-h - \alpha = \sigma$, the equation $[H\phi](\alpha) = \psi(\alpha)$ on $[-h, -h + \tau[$ becomes

$$A_h\phi(\sigma) + \int_{\sigma}^{0} A_{01}(\xi - h - \sigma)\, \phi(\xi)\, d\xi = \psi(-h - \sigma), \qquad \sigma \in\,]-\tau, 0]. \tag{2}$$

Since A_h is invertible, we have a Volterra integral equation which is uniquely solvable on $]-\tau, 0]$ for any ψ in $L^2[-h, 0]$. Solving it we obtain ϕ in $]-\tau, 0]$.

For $\alpha \geqslant -h + \tau$ we have

$$[H\phi](\alpha) = A_h\phi(-h - \alpha) + \int_{-h}^{-h+\tau} A_{01}(\theta)\, \phi(\theta - \alpha)\, d\theta + \int_{[-h+\tau, \alpha[} d\eta(\theta)\, \phi(\theta - \alpha)$$

a.e. in $[-h + \tau, 0]$.

Suppose that $\alpha \in [-h + m\tau, -h + (m+1)\tau[$ so that $\sigma = -h - \alpha \in]-(m+1)\tau, -m\tau]$, and that ϕ has been determined on $]-m\tau, 0]$. Now the equation $H\phi = \psi$ restricted to α, σ as indicated takes the form

$$A_h\phi(\sigma) + \int_\sigma^{\sigma+\tau} A_{01}(\xi - h - \sigma)\,\phi(\xi)\,d\xi + \int_{[\sigma+\tau,0[} d\eta_\xi(\xi - h - \sigma)\,\phi(\xi)$$
$$= \psi(-h - \sigma)$$

or, for almost all σ in $](m-1)\tau, m\tau]$,

$$A_h\phi(\sigma) + \int_\sigma^{-m\tau} A_{01}(\xi - h - \sigma)\,\phi(\xi)\,d\xi$$
$$= \psi(-h-\sigma) - \int_{-m\tau}^{\sigma+\tau} A_{01}(\xi - h - \sigma)\,\phi(\xi)\,d\xi + \int_{[\sigma+\tau,0[} d\eta_\xi(\xi - h - \sigma)\,\phi(\xi). \tag{3}$$

By hypothesis the right-hand side of (3) is a known function, and the left-hand side represents an invertible operator acting on the restriction of ϕ to $]-(m+1)\tau, -m\tau]$. This is again a Volterra integral equation which can be uniquely solved, and so we obtain ϕ on $]-(m+1)\tau, -m\tau]$ from the knowledge of ϕ on $]-m\tau, 0]$. Proceeding in this way to the left-hand side of the interval $[-h, 0]$ we obtain ϕ. ∎

Appendix to Section 3

Proof of Lemma 3.2. We first show that for all ϕ in $H_0^1[-h, 0]$

$$DH\phi = -HD\phi, \qquad D = d/d\theta. \tag{1}$$

We compute the distributional derivative $DH\phi$ of $H\phi$. For all ψ in $\mathscr{D}(]-h, 0[)$, the space of infinitely differentiable with compact support contained in $]-h, 0[$,

$$\langle DH\phi, \psi\rangle = -\int_{-h}^{0} d\alpha\, \frac{d\psi}{d\alpha}(\alpha) \cdot \int_{[-h,\alpha[} d\eta(\theta)\,\phi(\theta - \alpha),$$

where $\langle\ ,\ \rangle$ momentarily denotes the duality pairing between $\mathscr{D}'$ and $\mathscr{D}$. Using Fubini's theorem we change the order of integration:

$$\langle DH\phi, \psi\rangle = -\int_{[-h,0[} \int_\theta^0 d\alpha\, \frac{d\psi}{d\alpha}(\alpha) \cdot d\eta(\theta)\,\phi(\theta - \alpha).$$

Integrate the inner integral by parts

$$\langle DH\phi, \psi\rangle$$
$$= -\int_{[-h,0[} \left\{-\int_\theta^0 d\alpha\, \psi(\alpha)\cdot d\eta(\theta)\frac{d\phi}{d\alpha}(\theta-\alpha) + [\psi(\alpha)\cdot\eta(\theta)\,\phi(\theta-\alpha)]_{\alpha=\theta}^{\alpha=0}\right\}.$$

But $\psi(0) = 0$ and $\phi(0) = 0$ and boundary terms disappear. We change the order of integration once more to obtain

$$\langle DH\phi, \psi\rangle = -\int_{-h}^0 d\alpha\, \psi(\alpha)\cdot \int_{[-h,\alpha[} d\eta(\theta)\, D\phi(\theta-\alpha) = -(\psi, HD\phi)_2 .$$

This shows that the distributional derivative $DH\phi$ belongs to $L^2[-h, 0]$ and coincides with $-HD\phi$. This establishes (1). Finally

$$\| H\phi \|_{H^1}^2 = \| H\phi \|_2^2 + \| DH\phi \|_2^2 = \| H\phi \|_2^2 + \| HD\phi \|_2^2 \leqslant \| H \|^2 \| \phi \|_{H^1}^2 .$$

The lemma now follows directly from Theorem 2.1(ii) and (1). ∎

Proof of Theorem 3.6.

(i) $[S(t)][\phi] - [S(s)][\phi] = [S(t)\phi] - [S(s)\phi] = [S(t)\phi - S(s)\phi]$

and, by definition of the quotient norm,

$$\|[S(t)][\phi] - [S(s)][\phi]\| \leqslant \| S(t)\phi - S(s)\phi \|$$

and we can conclude that $[S(t)]$ is a strongly continuous semigroup on $M^2/\operatorname{Ker} F$. By definition $[\phi] \in \mathscr{D}([A])$ if and only if

$$\lim_{t\searrow 0} \frac{[S(t)][\phi] - [\phi]}{t} \text{ exists in } M^2/\operatorname{Ker} F.$$

But this is equivalent to saying that for all ϕ in $[\phi]$

$$\lim_{t\searrow 0} F\left(\frac{S(t)\phi - \phi}{t}\right) \text{ exists in } M^2.$$

It is readily seen that $[\mathscr{D}(A)] \subset \mathscr{D}([A])$. Since for all $t \geqslant 0$, $FS(t) = S^\top(t)^*F$, the above relation yields that

$$\lim_{t\searrow 0} \frac{S^\top(t)^*F\phi - F\phi}{t} \text{ exists in } M^2.$$

But this is true if and only if $F\phi \in \mathscr{D}(A^{\top *})$, where $A^{\top *}$ is the infinitesimal generator of the semigroup $\mathscr{S}^{\top *} = \{S^\top(t)^*: t \geqslant 0\}$. By definition

$$\mathscr{D}(A^{\top *}) = \{(\psi^0, \psi^1) \mid \psi^1 = H\hat{\psi}^0 + g, g \in H^1[-h, 0], g(-h) = 0\}.$$

Thus

$$F\phi \in \mathscr{D}(A^{\mathrm{T}*}) \Leftrightarrow H\phi^1 = H\hat{\phi}^0 + g \text{ (or } H(\phi^1 - \hat{\phi}^0) = g) \\ \text{for some } g \in H^1[-h, 0], g(-h) = 0.$$

But by Lemma 3.2, $H(\phi^1 - \hat{\phi}^0) = g \in H_h{}^1[-h, 0]$ means that there exists some ζ in $H_0{}^1[-h, 0]$ such that

$$H\zeta = g = H(\phi^1 - \hat{\phi}^0).$$

By construction $\xi = (\phi^0, \phi^0 + \zeta)$ belongs to $\mathscr{D}(A)$ and

$$F\xi = (\phi^0, H(\hat{\phi}^0 + \zeta)) = (\phi^0, H\hat{\phi}^0 + H(\phi^1 - \hat{\phi}^0)) = (\phi^0, H\phi^1) = F\phi.$$

This means that $[\phi] = [\xi] \in [\mathscr{D}(A)]$ and establishes that $\mathscr{D}([A]) \subset [\mathscr{D}(A)]$. Finally by definition for $[\phi]$ in $\mathscr{D}([A])$ there exists $\phi \in \mathscr{D}(A)$ such that $\phi \in [\phi]$

$$[A][\phi] = \lim_{t \searrow 0} \frac{[S(t)][\phi] - [\phi]}{t} = \lim_{t \searrow 0} \frac{[S(t)\phi - \phi]}{t} = \left[\lim_{t \searrow 0} \frac{S(t)\phi - \phi}{t}\right] = [A\phi].$$

(ii) By Proposition 3.1, for all $t \geqslant 0$, $S(t)^*F^* = F^*S^{\mathrm{T}}(t)$. Thus for all ψ in $\operatorname{Im} F^*$ there exists a ζ in M^2 such that $\psi = F^*\zeta$ and

$$S(t)^*\psi = S(t)^*F^*\zeta = F^*S^{\mathrm{T}}(t)\zeta \in \operatorname{Im} F^*;$$

moreover for all ψ in $\overline{\operatorname{Im} F^*}$ there exists a sequence $\{\zeta_n\}$ such that $F^*\zeta_n \to \psi$ and

$$S(t)^*\psi = \lim_{n\to\infty} S(t)^* F^*\zeta_n = \lim_{n\to\infty} F^*S^{\mathrm{T}}(t)\, \zeta_n \in \overline{\operatorname{Im} F^*}.$$

Thus the family of operators $S_F^*(t)$ obtained by restricting $S(t)^*$ to $\overline{\operatorname{Im} F^*}$ form a strongly continuous semigroup in $\mathscr{L}(\overline{\operatorname{Im} F^*})$.

(iii) By definition for all l in $(M^2/\operatorname{Ker} F)'$ and $[\phi]$ in $M^2/\operatorname{Ker} F$

$$[[S(t)]^*l, [\phi]] = [l, [S(t)\phi]] = ((\Lambda^{-1}l, S(t)\phi)) = ((S(t)^*\Lambda^{-1}l, \phi)).$$

But $\Lambda^{-1}l \in \overline{\operatorname{Im} F^*}$ and so necessarily $S(t)^*\Lambda^{-1}l \in \overline{\operatorname{Im} F^*}$. Therefore

$$[[S(t)]^*l, [\phi]] = ((S(t)^* \Lambda^{-1}l, \phi)) = ((S_F^*(t)\, \Lambda^{-1}l, \phi)) = [\Lambda S_F^*(t)\, \Lambda^{-1}l, [\phi]] \\ \Rightarrow [S(t)]^* = \Lambda S_F^*(t)\, \Lambda^{-1} \Rightarrow \Lambda^{-1}[S(t)]^*\Lambda = S_F^*(t). \quad \blacksquare$$

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